Flexural Vibrations of a Propped Cantilever

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The equations for the flexural vibrations of a propped cantilever beam have been used to compute a number of the vibration characteristics of such beams for the first five modes over the range of prop locations from 50 to 100 per cent of the length. Plots of these characteristics are included in the paper. This material has been prepared primarily for use in studies of contact spring vibration, and such application is briefly discussed. The mathematical treatment used to obtain the relations given is outlined in an Appendix.

I. INTRODUCTION

In relays and other switching apparatus, contact chatter and certain types of wear are associated with vibrations of the contact springs. As an aid in the study of these vibrations, the general theory of beam vibration has been used to develop an analytical treatment applicable to the important class of contact springs which can be considered as propped cantilever beams of uniform cross-section.

In almost all common types of switching apparatus, the contact springs are cantilever beams, clamped at the terminal end, which carry the contact at the free end. When the contact is open, the spring is usually propped or supported by a card or stud, and is therefore a propped cantilever. In some devices, the spring is supported at both the stud and contact when the latter is closed, and is then a doubly propped cantilever. In others, the spring is supported only at the contact when the latter is closed, and is therefore a singly propped cantilever in both operate and release, although the prop location differs for the two cases. Sometimes the mating contact is mounted on another spring, which constitutes a flexible prop, as contrasted with the (relatively) fixed and rigid prop provided by a card or stud.

The relations given here apply only to a uniform cantilever with a

single rigid prop. Some contact springs rigorously conform to these limitations, and the treatment is approximately applicable to a much larger number of cases. A more general treatment of relay spring vibration is given in Chapter 7 of Ref. 1. This includes an outline of approximate methods of analysis applicable to nonuniform springs and to those which are doubly propped, or supported by a compliant prop, such as a spring-mounted mating contact. The treatment given here may be used in applying these more general methods, but the present discussion is confined to the cases where it is directly applicable.

1.1 *Application to Chatter Studies*

The contact chatter of primary interest is that occurring with closed contacts, usually immediately following closure. With a fixed mating contact, the moving spring is a cantilever propped at the contact. Vibration results in modulation of the contact force and therefore of the contact resistance. If the amplitude of the force modulation exceeds the static contact force, a transient open occurs. The timing and duration of these opens can therefore be related to the force modulation and to the amplitudes and frequencies of the spring vibrations. The latter may be directly observed, or predicted from an analysis of the excitation of this vibration involved in operation.

1.2 *Application to Wear Studies*

The wear associated with vibration may occur at the contact or at a supporting or actuating card or stud which serves as the prop to a contact spring. Relative motion in the direction of the spring length results in wear. Severe wear occurs when such longitudinal motion is imposed in actuation. When this is avoided by providing purely perpendicular motion in actuation, wear may still be produced by the longitudinal component of the vibratory motion. The relations given here include those between the longitudinal amplitudes and the (normal) displacement amplitudes, or the corresponding energy content. Thus the longitudinal amplitude can be evaluated from the observed displacement amplitude, or from the estimated energy content of the spring vibration.

II. THEORETICAL FOUNDATION

The equations giving the spring vibration characteristics are derived in the Appendix to this paper. The treatment follows the usual approximate theory of beam vibration, based on the simple theory of bending,

and formally applicable only for displacements which are small compared with the spring dimensions. These formal limitations are of little consequence for the present purpose, although impact causes some deformation other than simple bending.

For springs of uniform section, flexural vibrations conform to a general differential equation (3),* having a solution of the form of (4). This represents a harmonic motion in which all points in the spring move in phase. The relative motion at different points is determined by the dynamic deflection curve X, a function of x only, where x is measured along the length of the spring. Each such deflection curve corresponds to a particular mode of vibration, having a corresponding characteristic frequency. The deflection curves for the first three modes of a propped spring (prop at 85 per cent of the length) are shown in Fig. 1. As there indicated, several modes may be present together, resulting in a configuration which is at any instant the sum of the different modes present.

The deflection curves for the several modes, and the corresponding frequencies, depend upon boundary conditions determined by the way in which the spring is supported. For a propped cantilever, the boundary conditions, and hence the deflection curves and frequencies, vary with the prop location (defined by the ratio L'/L of Fig. 1). The special cases in which L'/L is zero and unity correspond respectively to a free cantilever and an end-propped cantilever. All the relations given in the figures are shown in the form of curves in which the quantity given is plotted against L'/L over the range from 0.5 to 1.0, which covers the prop locations applying to most contact springs.

The frequency equation for any particular prop location is transcendental in form (8). The successive roots of this equation determine the frequencies and deflection curves of the several modes. These roots do not form a simple series, and the successive frequencies are not simple multiples of the fundamental. In the higher modes, however, the deflection curves approach sine curves in form (except for the end sections), and the intervals between successive frequencies are approximately equal.

From the frequency constant for a particular mode and prop location there may be determined all the constants of the corresponding deflection curve except for an undetermined multiplier (A_1 in the equations of the Appendix), which measures the amplitude or energy content T of the mode in question. As this constant determines both the energy content and the maximum deflection (or amplitude) at any point on the beam,

* Equations are cited by the numbers identifying them in the Appendix.

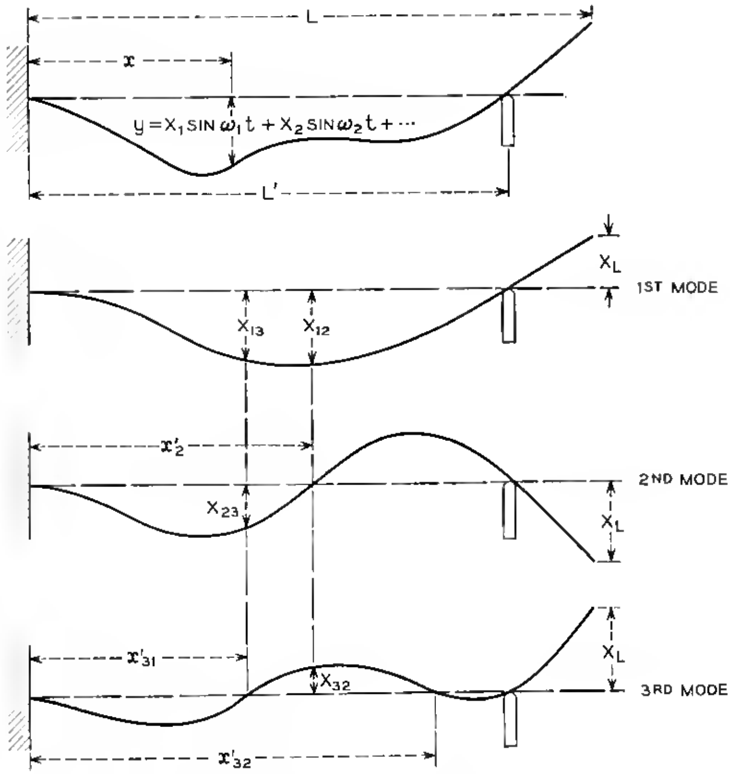

Fig. 1 — Flexural vibrations of a propped cantilever.

it may be eliminated from the equations to express the amplitude at specified points on the beam in terms of the energy content. Similarly, this amplitude constant may be eliminated from expressions for the force acting on the prop and for the longitudinal displacement there to give expressions for these quantities in terms of the energy content.

III. FREQUENCY RATIOS

The frequencies of the first five propped modes are shown in Fig. 2. These frequencies are given as multiples of f_0, the frequency of the same beam as a free cantilever, and are shown plotted against the prop location as measured by L'/L.

The reference frequency f_0 is given by equation (12):

$$f_0 = 0.323 \sqrt{(s/m)},$$

where s is the static stiffness of the beam and m is its actual mass. Equivalent expressions are the following:

For a circular section: $$f_0 = \frac{0.1416d}{L^2}\sqrt{\frac{E}{\rho}},$$

For a rectangular section: $$f_0 = \frac{0.1636t}{L^2}\sqrt{\frac{E}{\rho}},$$

where d is the diameter of the circular section, t the thickness of the rectangular section, L is the length, and $\sqrt{E/\rho}$ is the velocity of sound in the material.

The frequencies given by these relations apply to springs of uniform cross-section. The added mass of the contact in relay springs reduces the frequency (except when propped at the contact). An approximate correction for the effect of the contact may be made by determining the effective mass m' of the spring for the mode in question by the procedure given in Section V. Then if m'' is the mass of the contact, the corrected frequency is the product of $\sqrt{m'/(m' + m'')}$ and the frequency read from Fig. 2.

IV. LOCATIONS OF NODES AND LOOPS

A node is a point of zero displacement (other than the prop location), while a loop is a point of maximum displacement. As illustrated in Fig. 1, the number of loops is the same as the order of the mode, while the number of nodes is one less than the order of the mode. Expressions for determining the locations of the nodes and loops are given in the Appendix.

Fig. 3 gives the locations of the nodes of the second and third modes.

V. RELATIONS OF AMPLITUDES TO ENERGY CONTENT

For any particular mode, the amplitude at any specified point on the spring is determined by the energy content T. Thus an estimate of T may be used to estimate the amplitude at some specified point, or the observed amplitude may be used to determine the energy content.

Fig. 4 gives the relation between the energy content T and the amplitude X_L of the free end of the spring, expressed as values of the ratio $m\omega^2 X_L^2/T$, where $\omega/(2\pi)$ is the frequency and m is the total mass of the spring. Even when a correction is made for contact mass in determining the frequency, $m\omega^2$ should be taken as the product of the mass

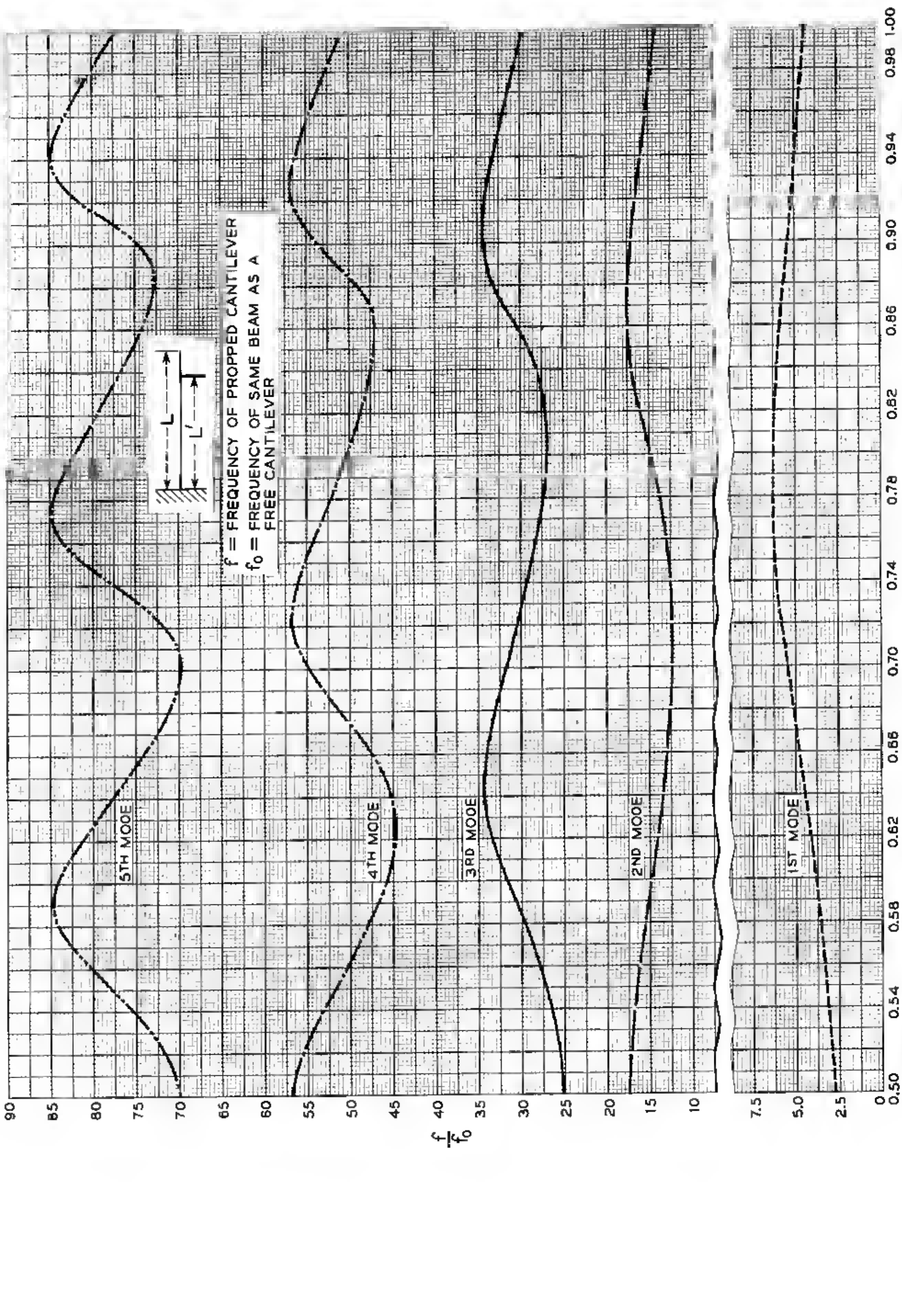

Fig. 9 — Frequency of a propped cantilever.

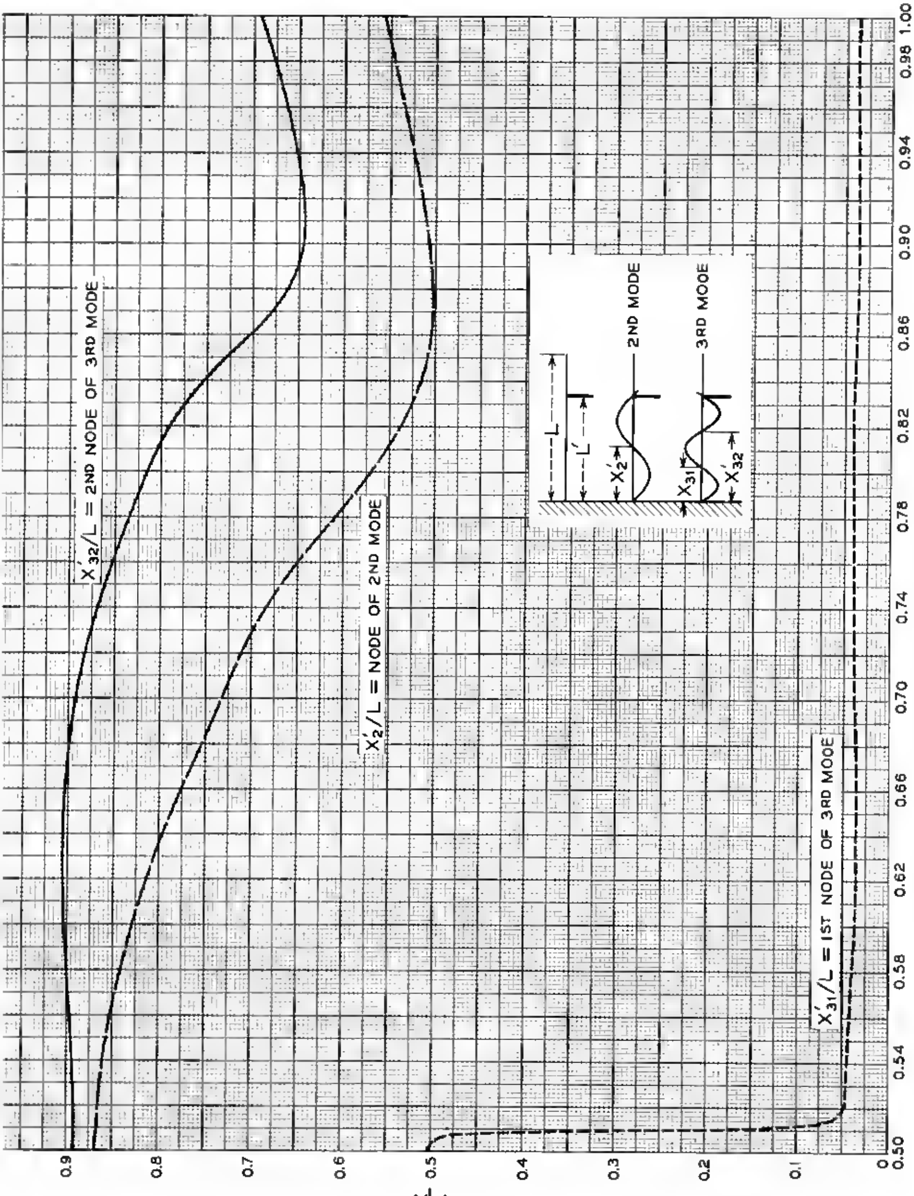

Fig. 3 — Propped cantilever: nodes of 2nd and 3rd modes.

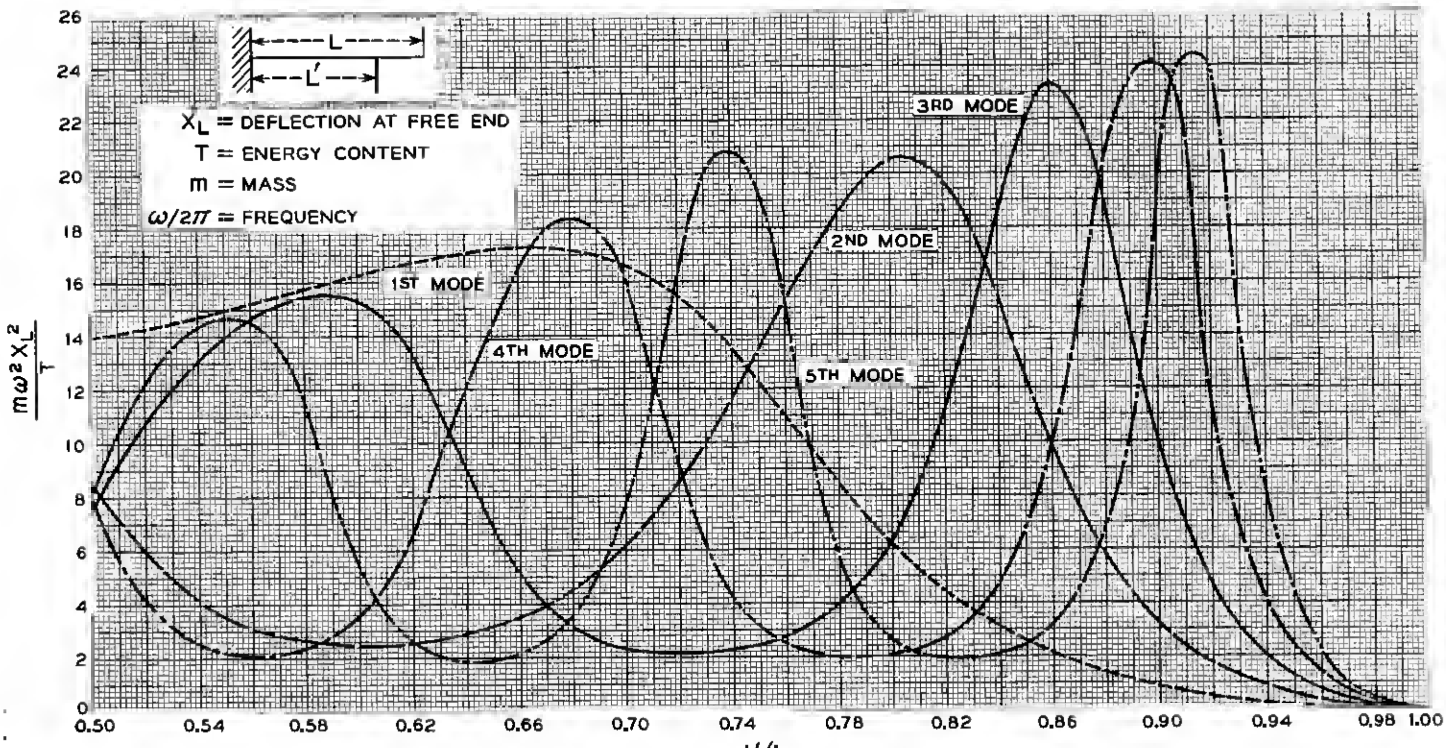

Fig. 4 — Propped cantilever: energy content in terms of end amplitude.

of the spring alone and the uncorrected frequency, without allowance for contact mass.

The effective mass m' of the spring, in terms of end motion, is the kinetic energy T divided by half the square of the end velocity ωX_L. Thus the ratio m'/m is twice the reciprocal of $m\omega^2 X_L^2/T$, given in Fig. 4, and values read from this figure may be used to evaluate m'.

These curves may be used to determine the energy content from observations of the end amplitude. When the prop is close to the free end, the end amplitude is smaller than that at or near the loops. When two or more modes are present, it is convenient to measure the amplitude at the node of one of the modes present. Values have been determined, therefore, of the ratios $m\omega_1^2 X_{12}/T$, $m\omega_1^2 X_{13}/T$, $m\omega_2^2 X_{23}/T$, and $m\omega_3^2 X_{32}/T$, which are given in Fig. 5. As shown in Fig. 1, X_{12} and X_{32} are the amplitudes of the first and third modes respectively at the node (x_2') of the second mode, while X_{13} and X_{23} are the amplitudes of the first and second modes respectively at the rear node (x_{31}') of the third mode. The locations of these nodes (x_2' and x_{31}') are given in Fig. 3.

When two or more modes are present and it is desired to determine the energy contents of the separate modes, the separate amplitudes must first be determined. This requires measuring the displacements at successive time intervals and using these successive displacements in a set of equations which can be solved for the amplitudes. If it can be assumed, for example, that only the first three modes are present, then the displacement X at a node (such as x_{31}') of the third mode is the sum of the first two modes, and is given by:

$$X = X_{13} \sin (\omega_1 t + \varphi_1) + X_{23} \sin (\omega_2 t + \varphi_2),$$

where φ_1 and φ_2 are the (unknown) phase angles of the two modes with respect to an arbitrary choice of the time origin. Let X_1 be the observed value of X at this selected time origin, and let X_2, X_3 and X_4 be the observed values of X at the times at which $\omega_1 t$ is equal to $\pi/2$, π, and $3\pi/2$, respectively. On substituting these corresponding values of X and t in the preceding equation, there are obtained four equations in the four unknowns: $X_{13} \sin \varphi_1$, $X_{13} \cos \varphi_1$, $X_{23} \sin \varphi_2$, and $X_{23} \cos \varphi_2$. These four unknowns may be evaluated from the determinant D given by:

$$D = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & \sin a_1 & \cos a_1 \\ 0 & -1 & \sin a_2 & \cos a_2 \\ -1 & 0 & \sin a_3 & \cos a_3 \end{vmatrix}$$

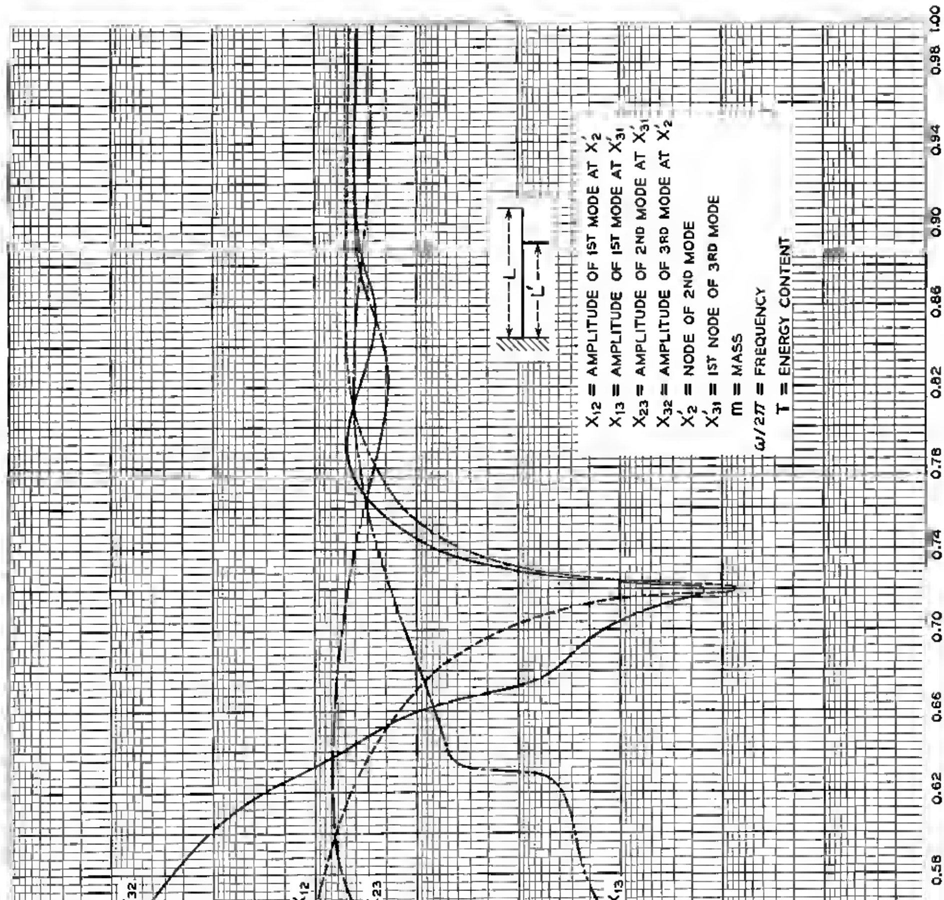

Fig. 5 — Propped cantilever: energy content in terms of amplitudes at specific locations

where:
$$a_1 = \frac{f_2}{f_1}\cdot\frac{\pi}{2}, \qquad a_2 = \frac{f_2}{f_1}\cdot\pi, \qquad a_3 = \frac{f_2}{f_1}\cdot\frac{3\pi}{2}.$$
The ratio f_2/f_1 can be read from Fig. 2.

The same procedure may be employed to determine the values of X_{12} and X_{32} from observations of the deflections at the node x_2' of the second mode. (The same equations apply in this case, except that f_2/f_1 in the determinant terms is replaced by f_3/f_1.) A check on the accuracy of the computations (or of the assumption that only the first three modes are present) is given by comparing the values for the energy content of the first mode obtained (by means of Fig. 5) from the values found for X_{12} and X_{13}: these values of T should be the same.

VI. FORCE AT PROP

A prop, or point of simple support, is taken as restraining the beam from deflection, without the application of any moment (or clamping action). Aside from the minor variation in the instantaneous point of support resulting from the finite dimensions of supporting surfaces, this condition is satisfied by the support actually provided when a spring is tensioned against a stud or contact. In general, vibration results in a force modulation $F' \sin \omega t$ corresponding to each mode present, where $\omega/(2\pi)$ is the frequency of that mode and F' is proportional to the square root of its energy content. As the sense of this force modulation varies with the phase of the mode, it alternately increases or decreases the total tension against the prop, which includes the static tension and the total force modulation of all modes present. The propped mode equations only apply rigorously when this total tension has the same sense as the static tension, as otherwise the prop is no longer effective and the spring moves away from it. (Practically, the effect of such separation may be ignored if it occurs only over a short interval of time.)

This force modulation is related to contact chatter, contact noise, and wear. When the spring is propped at the contact, an open results whenever the total tension becomes zero (or small enough to produce high contact resistance). Similarly, contact resistance variations resulting from force variations produce noise. Wear, whether at a contact or other support, such as a card, depends upon both the longitudinal motion and upon the normal force, or total tension.

If the energy content of a mode is estimated, or determined from amplitude observations, the amplitude of the force modulation for the first three modes may be determined from Fig. 6. This gives values of

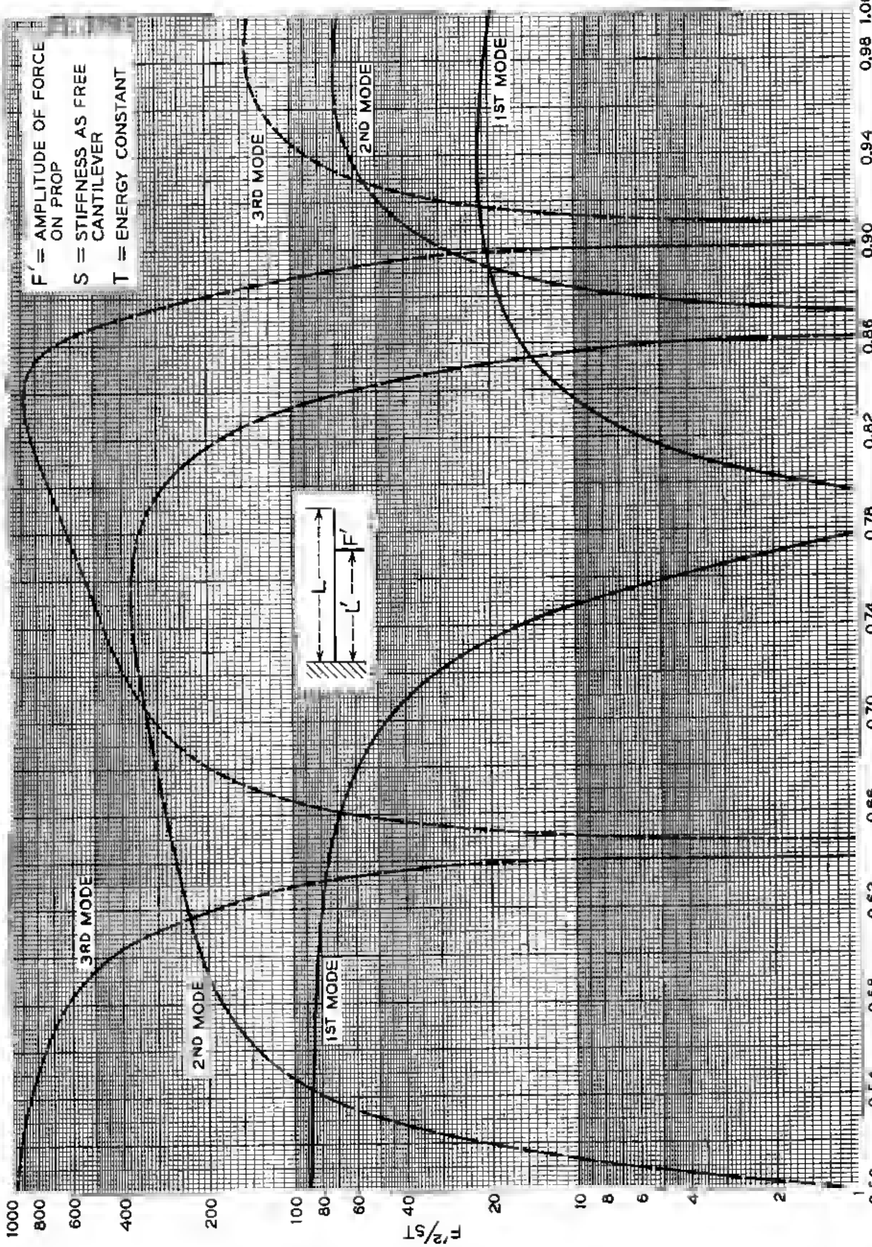

Fig. 6 — Propped cantilever: force on prop.

$F'^2/(sT)$, where T is the energy content of the mode in question and s is the static stiffness of the spring. The latter quantity may be measured directly, or computed as $3EI/L^3$, where E is Young's modulus for the spring material and I is the moment of inertia of its cross-section.

It will be seen from Fig. 6 that F'^2/sT varies greatly in magnitude with the prop location. Except for the fundamental, all modes have prop locations for which F' is zero. These correspond to the nodes of corresponding modes of a free (unpropped) cantilever, in which the spring vibrates without deflection at the prop without requiring any restraint, and hence without force modulation.

In general, and in particular for an end prop, as with a closed contact, F'^2/sT increases with the order of the mode. Hence a given energy content produces a greater force modulation, and is therefore more likely to produce chatter, the higher the order of the mode. (The open intervals, on the other hand, are necessarily shorter with the higher modes, because of the higher frequencies.) In relay spring vibration, the energy content of the fundamental is usually larger than that of the other modes, so that chatter commonly occurs at intervals equal to the period of the fundamental, but each interval may comprise a number of brief opens, corresponding to the shorter intervals in which higher modes are in phase with the fundamental. The fine chatter immediately following contact impact, however, corresponds wholly to higher modes, occurring at a time when the sense of the fundamental force modulation is the same as that of the static force.

VII. LONGITUDINAL COMPONENT OF VIBRATORY MOTION

In vibration, the deflected position of the spring defines a path from the clamp to the prop point which is necessarily longer than the distance between these points measured along the rest position of the spring. The difference between these two lengths represents a longitudinal component of the motion at the prop. This may be termed the vibratory slide, as distinguished from the slide resulting, for example, from motion of the prop point in actuation. The amplitude of this motion is of interest in connection with wear, particularly the wear of a card serving as a prop. The vibration characteristics affecting the wear are the amplitude of this vibratory slide, and the normal force on the prop, which varies with the force modulation discussed in the preceding section.

The longitudinal displacement is zero for the rest position of the spring and attains full amplitude, or maximum displacement, for full amplitude of the normal deflection in either sense. The longitudinal motion there-

fore has twice the frequency of the flexural vibration producing it. As shown in the Appendix, the longitudinal motion is harmonic, and as such has an amplitude $Z/2$ about a displacement $Z/2$ from the rest position, where Z is the full excursion, or double amplitude.

When a single mode is present alone, the longitudinal motion has only one component, with a frequency twice that of the mode producing it. Values of the ratio $m\omega^2L'Z/T$ have been determined for the first five modes, and are given in Fig. 7. It will be seen that the values of this ratio increase with the order of the mode. To obtain comparable values of Z for the same energy content T, however, these values of $m\omega^2L'Z/T$ must be divided by $(f/f_1)^2$, and when this is done it is found that the longitudinal displacement for the same energy content decreases with the order of the mode.

If two or more modes are present together, the longitudinal motion includes the motion that either would produce separately, and additional motion at frequencies, as shown in the Appendix, equal to the sums and differences of the frequencies of the modes present. Expressions for the amplitudes of the additional motions are included in the Appendix, and these were evaluated for the case where the first and second modes only are present. The additional displacement was found to be only five per cent of that produced by the first mode alone (for equal energy contents of the two modes). Thus the longitudinal motion when two or more modes are present differs little from the sum of the motions that each would produce separately for the same energy content.

For a given total energy content, therefore, the longitudinal amplitude is a maximum when only the fundamental mode is present. This, however, does not suffice to show that the wear is a maximum if all the energy is in the fundamental, rather than distributed among several modes. The wear also varies with the normal force, and it was shown in the preceding section that the force modulation for a given energy content is greater the higher the mode. It would therefore be necessary to know the relation of wear to both longitudinal motion and normal force to determine how the wear varies with the distribution of energy among the possible modes.

VIII. CONTACT WIPE

There is another type of longitudinal motion that occurs at a closed contact (end propped spring) because the contact surface is offset from the center line of the spring by a distance L''. This results in a longitudinal motion $z' = Z' \sin \omega t$ at the contact surface, where $Z' =$

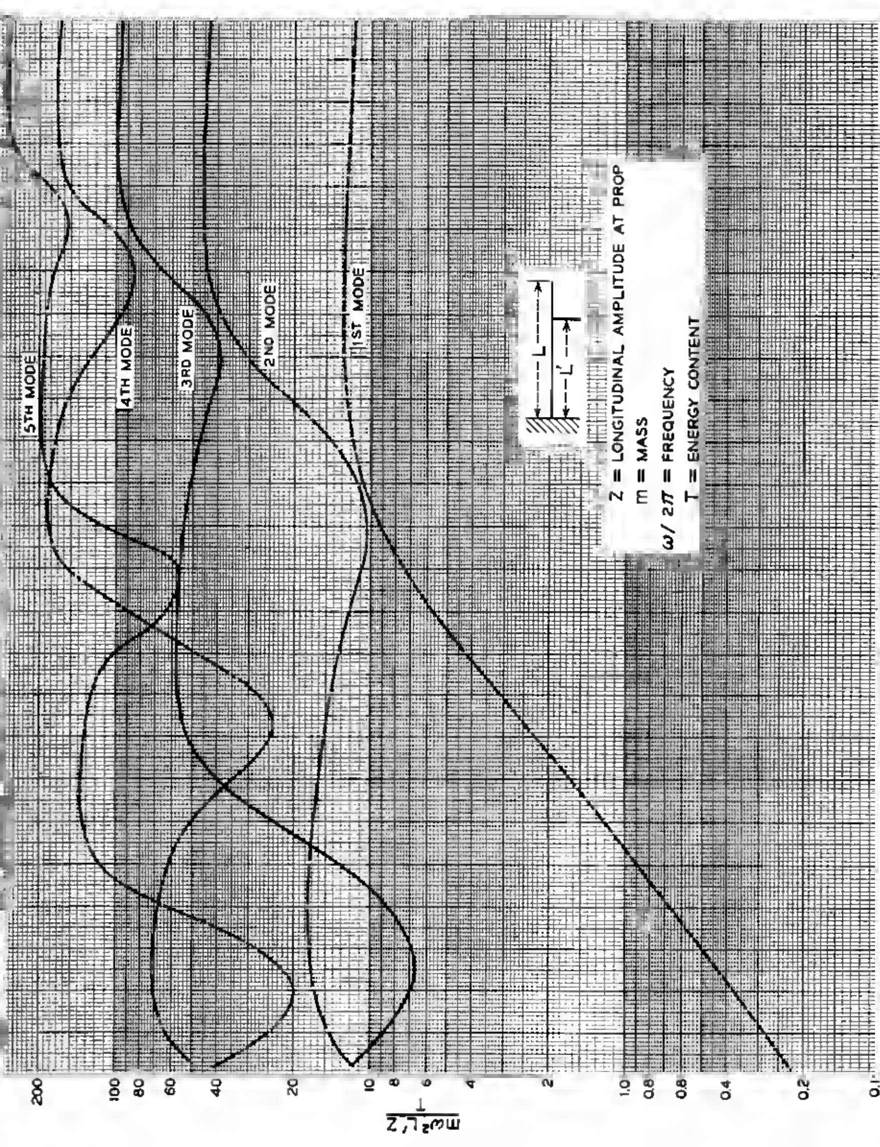

Fig. 7 — Propped cantilever: slide amplitude at prop.

$L''(dX/dx)_L$. This quantity can be evaluated from the table of the ratio $m\omega^2L^2(dX/dx)_L^2/T$ given Section A.10 of the Appendix.

IX. DISCUSSION

The material contained in this paper has been prepared for reference purposes and for use in analytical or experimental studies, primarily those relating to contact springs (although it is of course applicable to any use of propped cantilever beams). Few conclusions of engineering interest may be drawn directly from this material; such value as it has must appear in its application. The possible use of the material may be indicated here by a brief discussion of its application to relay spring vibration.

Relay operation results in spring vibration, and such vibration may result in contact chatter and also in wear, particularly of such actuating members as studs or cards. The vibration amplitudes and frequencies (at least of the lower modes) are readily observed and measured, for example by the rapid record shadowgraph.[2] To reduce chatter and wear, information is required as to (a) the relations between the vibration characteristics and the relay design and conditions of actuation, and (b) the relations between chatter and wear and the vibration characteristics.

Such information may be obtained either by analysis or by direct experiment, but in either case the vibration characteristics are involved. In studying the excitation problem analytically, the amplitudes and frequencies must be determined from energy estimates, while an experimental study requires that the energy be evaluated from observed amplitudes.

The incidence of chatter can be determined directly from knowledge of the vibration, provided the force modulation is computed from the observed amplitudes by the relations given here. A similar analysis of wear would require knowledge of the dependence of wear (for particular materials) on both normal force and longitudinal displacement. Information as to these relations is incomplete, but if available their application would require the determination from the observed vibration of the resulting normal force variation and longitudinal displacement by means of the relations given here.

Because of the relatively large amplitudes associated with the fundamental mode, it is the most conspicuous feature of relay spring vibration. The relation given above between the normal force and the energy content shows that the force modulation for a given energy increases

with the order of the mode, indicating the considerable effect on chatter that may result from the presence of higher modes. The effect of the higher modes on wear is less well understood. The relations given here show that the diversion of energy from lower to higher modes increases the force modulation, but decreases the longitudinal displacement.

APPENDIX

Derivation of Equations

A.1 *Equations of Flexural Vibration*

As shown in such texts as Ref. 3, bending of a beam may be described in terms of the deflection y at a point located at a distance x, measured along the length of the beam from a clamp or other point of reference. Then dy/dx is the slope of the deflection curve, and, for the small deflections assumed in the simple theory of bending, the curvature of the neutral axis is given by d^2y/dx^2. In this simple theory, the moment M at the point x is given by:

$$M = EI(d^2y/dx^2), \tag{1}$$

where E is Young's modulus and I is the moment of inertia of the beam's cross-section. The shearing force F is equal to dM/dx, and is therefore given by:

$$F = EI(d^3y/dx^3). \tag{2}$$

In motion, the inertia reaction of a differential element of length must equal the difference between the shearing forces at the ends of the element. Hence:

$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho\alpha}\frac{\partial^4 y}{\partial x^4} = 0, \tag{3}$$

where ρ is the density of the beam and α is the area of its cross-section. Equation (3) is the general differential equation for flexural vibrations of beams of uniform section, assuming the simple theory of bending to apply. The general form of solution is given by:

$$y = X \sin(\omega t + k), \tag{4}$$

where X is a function of x only, the solution to the equation $d^4X/dx^4 = c^4X$, given by

$$X = A(\sin cx + B\cos cx + C\sinh cx + D\cosh cx), \tag{5}$$

in which c is given by:

$$c^4 = \omega^2 \rho \alpha / EI. \tag{6}$$

A.2 *Modes of a Propped Cantilever*

For a propped cantilever, x may be measured from the clamped end, as indicated in Fig. 1. Let L be the length of the beam, and L' the value of x for the prop. Subject to continuity, different forms of (5) apply at either side of the prop: let these be X_1 for $x < L'$, and X_2 for $x > L'$, and write (5) as:

$$X_1 = A_1 (\sin cx + B_1 \cos cx + C_1 \sinh cx + D_1 \cosh cx), \tag{5a}$$

$$X_2 = A_2 (\sin c(x - L') + B_2 \cos c(x - L') + C_2 \sinh c(x - L') + D_2 \cosh c(x - L')). \tag{5b}$$

Writing X', X'', X''' for the successive derivatives of X with respect to cx, the boundary conditions applying are as follows:

$$\begin{aligned} &\text{For } x = 0, \quad X_1 = X_1' = 0, \\ &\text{For } x = L', \quad X_1 = X_2 = 0, \quad X_1' = X_2', \quad X_1'' = X_2'', \\ &\text{For } x = L, \quad X_2'' = X_2''' = 0. \end{aligned} \tag{7}$$

Writing a for cL', and b for $c(L - L')$, substitution of the expressions for X_1 and X_2 in the boundary conditions (7) gives the frequency equation:

$$\frac{\cos a \cdot \sinh a - \sin a \cdot \cosh a}{1 - \cos a \cdot \cosh a} = \frac{\cos b \cdot \sinh b - \sin b \cdot \cosh b}{1 + \cos b \cdot \cosh b}, \tag{8}$$

and the following expressions for the coefficients:

$$\begin{aligned} C_1 &= -1, \\ B_1 &= -D_1 = -\frac{\sin a - \sinh a}{\cos a - \cosh a}, \\ C_2 &= \frac{1 + \cos b \cdot \cosh b - \sin b \cdot \sinh b}{1 + \cos b \cdot \cosh b + \sin b \cdot \sinh b}, \\ -B_2 = D_2 &= \frac{\sin b \cdot \cosh b - \cos b \cdot \sinh b}{1 + \cos b \cdot \cosh b + \sin b \cdot \sinh b}, \\ \frac{A_1}{A_2} &= \frac{\cos a - \cosh a}{1 - \cos a \cdot \cosh a} \cdot \frac{1 + \cos b \cdot \cosh b}{1 + \cos b \cdot \cosh b + \sin b \cdot \sinh b}. \end{aligned} \tag{9}$$

The ratio b/a ($= L/L' - 1$) is determined by the prop location. Then the values of a which satisfy (8) determine values of c ($= a/L'$), which correspond to values of the frequency $\omega/(2\pi)$ given by (6). The successive frequencies thus determined are those of successive modes of vibration. Substitution in (9) of the value of a for any particular mode determines the coefficients of the expression for the corresponding deflection curve. The one remaining coefficient, A_1 or A_2, measures the amplitude.

A.3 *The Frequency Equation*

As the mass m of the beam is equal to $\rho\alpha L$, (6) may be written in the form:

$$\omega^2 = (s/3m)(cL)^4, \tag{10}$$

where s is the static stiffness of the beam as a free cantilever, or $3EI/L^3$. As $a = cL'$, the values of a satisfying (8) for a given prop location, or value of L'/L, determine corresponding values of cL, and hence corresponding values of the frequency $\omega/(2\pi)$.

The special case in which $L' = 0$, or $a = 0$, $b = cL$, is that of a simple (unpropped) cantilever, and (8) then reduces to:

$$1 + \cos b \cdot \cosh b = 0. \tag{11}$$

The first three roots of this equation give values of b^2, or $(cL)^2$, of 3·52, 22·0 and 61·8. For the higher roots, a good approximation to b is given by $\pi(n - \frac{1}{2})$, where n is an integer. From (10), the frequency f_0 of the fundamental cantilever mode is given by:

$$f_0 = 0.323\sqrt{(s/m)}, \tag{12}$$

and the frequency of any other mode is given in terms of f_0 by:

$$\frac{f}{f_0} = (cL)^2/3.52 \tag{13}$$

For various values of L'/L, (8) has been solved numerically to determine the values of a and thus of cL for the first five modes. By means of (13), the resulting values of f/f_0 have been determined, and are plotted in Fig. 2.

Another special case of interest is that of an end prop, for which $L' = L$, or $a = cL$, $b = 0$. In this case, (8) reduces to:

$$\tan a = \tanh a. \tag{14}$$

The values of a (or cL) satisfying this equation are given approximately by $\pi(n + \frac{1}{4})$, where n is an integer. The value of $(cL)^2$ for the fundamental end propped mode is 15.50, giving a frequency 4.40 times f_0, the frequency of the fundamental cantilever mode.

A.4 *Nodes and Loops of Deflection Curves*

On substituting in (5) the expressions for the coefficients for X_1 given by (9), the deflection curve for $x < L'$ is given by:

$$X_1/A_1 = [f_1(a) - f_1(u)]\,(\cosh u - \cos u), \tag{15}$$

where $u = cx$, and $f_1(u)$ is given by:

$$f_1(u) = \frac{\sin u - \sinh u}{\cos u - \cosh u}. \tag{16}$$

Thus the nodes (points of zero deflection) lying between the clamp and the prop ($x < L'$) occur at those values of u for which $f_1(u) = f_1(a)$. These values of u may thus be determined for any mode and prop location from (16) and the corresponding value of a. As $u/a = x/L'$, there may thus be determined the values of the node locations x'/L lying between the clamp and prop. The locations of the nodes for the second and third modes lying in this region are plotted against L'/L in Fig. 3.

The loops (points of maximum deflection) of the deflection curve occur at those values of u for which $dX/du = 0$. By differentiation of (15), it is found that these values of u for $x < L'$ are those for which $f_2(u) = f_1(a)$, where $f_2(u)$ is given by:

$$f_2(u) = \frac{\cosh u - \cos u}{\sinh u + \sin u}. \tag{17}$$

For any mode and prop location and the corresponding value of a there can be determined those values of u for which $f_2(u) = f_1(a)$. From these can be determined the corresponding loop locations x/L lying between the clamp and plot.

Similarly, from (5) and the coefficients of X_2 given by (9), the node locations lying beyond the prop, $x > L'$, are given by:

$$0 = B_3 \sin (u - a) + B_4 \sinh (u - a) \\ - B_5\,[\cos (u - a) - \cosh (u - a)],$$

where:

$$B_3 = 1 + \cos b \cosh b + \sin b \sinh b,$$

$$B_4 = 1 + \cos b \cosh b - \sin b \sinh b,$$

$$B_5 = \sin b \cosh b - \cos b \sinh b.$$

The node locations in Fig. 3 lying beyond the prop have been determined from these equations.

Similarly, the loop locations lying beyond the prop are given by:

$$0 = B_3 \cos(u - a) + B_4 \cosh(u - a) + B_5 [\sin(u - a) + \sinh(u - a)].$$

A.5 *Free-End Deflection*

On substituting in (5) the expressions for the coefficients given by (9), it is found that the free-end deflection X_L, the value of X_2 for $x = L$, is given by:

$$\frac{X_L}{A_2} = \frac{2(\sin b + \sinh b)}{1 + \cos b \cdot \cosh b + \sin b \cdot \sinh b}, \tag{18}$$

or by:

$$\frac{X_L}{A_1} = \frac{2(\sin b + \sinh b)(1 - \cos a \cdot \cosh a)}{(1 + \cos b \cdot \cosh b)(\cos a - \cosh a)}. \tag{19}$$

A.6 *Energy Content*

The energy content of a vibrating beam is the integral over the length of $\rho\alpha\dot{y}^2 \cdot dx/2$, where $\dot{y}$, or dy/dt, is the maximum velocity (occurring at zero deflection, when all the energy is kinetic). Then, from (4), the energy content T is given by:

$$2T = \omega^2 \rho\alpha \int_0^L X^2 \cdot dx. \tag{20}$$

For the propped beam, the integral

$$\int_0^L X^2 \cdot dx$$

is given by:

$$\int_0^L X_1^2 \cdot dx + \int_{L'}^L X_2^2 \cdot dx.$$

As shown in Ref. 3 these two component integrals are given by:

$$
\begin{aligned}
4c \int_0^{L'} X_1^2 \cdot dx &= \Big|_0^{L'} [3X_1X_1''' - X_1'X_1'' \\
&\qquad + cx(X_1^2 - 2X_1'X_1''' + X_1''^2)] \\
4c \int_{L'}^{L} X_2^2 \cdot dx &= \Big|_{L'}^{L} [3X_2X_2''' - X_2'X_2'' \\
&\qquad + c(x - L')(X_2^2 - 2X_2'X_2''' + X_2''^2)].
\end{aligned} \tag{21}
$$

On substituting the boundary conditions given by (7), the expression for the energy content reduces to:

$$T = \frac{\omega^2 \rho \alpha}{8} [(L - L')X_L^2 + L'(X_{L'}''^2 - 2X_{L'}'X_{L'}''')], \tag{22}$$

where $X_{L'}'$, $X_{L'}''$ and $X_{L'}'''$ are the first three derivatives of X_1 with respect to cx at $x = L'$, and X_L is, as before, the value of X_2 at $x = L$. Expressions for the derivatives can be obtained by differentiation of (15), from which it is found that the second term in the bracket of (22) is equal to $4L'A_1^2f_1^2(a)$. Then (22) reduces to:

$$\frac{T}{m\omega^2 X_L^2} = \frac{1}{8}\left[1 - \frac{L'}{L} + 4\,\frac{L'}{L}\left(\frac{A_1}{X_L}\right)^2 f_1^2(a)\right], \tag{23}$$

where X_L/A_1 is given by (19), and $f_1(a)$ by (16). Values of $m\omega^2 X_L^2/T$ for the first five modes have been computed from (23) and are shown plotted against L'/L in Fig. 4.

As the effective mass m' in terms of motion at the free end is defined as the kinetic energy T divided by half the square of the end velocity, or $(\omega X_L)^2/2$, the quantity given by (23) is one half the ratio of the effective mass of the beam to its actual mass, or $m'/(2m)$.

A.7 *Energy Content in Terms of Amplitude at Nodes of Other Modes*

Let X_{12} be the deflection in the first mode at $x = x_2'$, the location of the node of the second mode. Using the value of a applying to the first mode (for a particular prop location L'/L), and taking $u = ax_2'/L'$, (15) may be used to determine X_{12}/A_1. From this and the value of $T/(m\omega^2 A_1^2)$ given by (23) there may be determined the value of $m\omega^2 X_{12}^2/T$. Such values have been determined for various values of L'/L, and are plotted against L'/L in Fig. 5.

The same procedure has been used to determine the values of $m\omega^2 X_{13}^2/T$, $m\omega^2 X_{23}^2/T$ and $m\omega^2 X_{32}^2/T$ given in Fig. 5. X_{32} is the amplitude of the third mode at x_2', while X_{13} and X_{23} are the amplitudes of

the first and second modes respectively at x_{31}', the first (rear) node of the third mode.

A.8 *Force Modulation at the Prop*

The shearing force F on a section of the beam is given by (2), and its maximum value, or force amplitude, is therefore given by $c^3EI \cdot X'''$ (where, as before, $X''' = d^3X/du^3$). The discontinuity at the prop results in a difference between X_1''' and X_2''' corresponding to a force $F' \sin(\omega t + k)$ acting on the prop, where F' is given by:

$$F' = c^3EI(X_1''' - X_2''')_{(x=L')} \tag{24}$$

The force $F' \sin(\omega t + k)$ is, of course, additive to any static tension acting at the prop.

Expressions for X_1''' and X_2''' at $x = L'$ may be obtained by differentiation of (5) and substitution of the expressions for the coefficients given by (9), giving:

$$\frac{F'}{A_1c^3EI} = f_3(b)\frac{X_L}{A_1} - 2\frac{f_3(a)}{f_2(a)}, \tag{25}$$

where $f_3(u)$ is given by:

$$f_3(u) = \frac{\sin u \cdot \sinh u}{\sin u + \sinh u}. \tag{26}$$

The quantity given by the left side of (25) may be squared and simplified as follows:

$$\left[\frac{F'}{A_1c^3EI}\right]^2 = \frac{L^3F'^2}{A_1^2EI(cL)^2} \cdot \frac{1}{EIc^4L} = \frac{3F'^2}{sA_1^2(cL)^2\omega^2m},$$

where, from (6), $\omega^2\rho\alpha$ is substituted for c^4EI, m is substituted for $\rho\alpha L$, and s for $3EI/L^3$. As before, m is the mass of the spring and s is its stiffness as a free cantilever.

It follows that $F'^2/(sT)$ is given by:

$$\frac{F'^2}{sT} = \frac{(cL)^2}{3} \cdot \left(\frac{F'}{A_1c^3EI}\right)^2 \cdot \frac{m\omega^2A_1^2}{T}. \tag{27}$$

Using (23) and (25) to evaluate the second and third terms on the right-hand side of (27), the latter equation has been used to determine values of $F'^2/(sT)$ for the first three modes for various prop locations (L'/L). The results are plotted in Fig. 6.

A.9 Longitudinal Component of Motion at the Prop

The longitudinal component of the motion at the prop is the displacement z in the direction of the spring length. This is equal to the difference between the distance from the clamp to the prop measured along the displacement curve and that measured along the rest position. As the latter distance is L', z is given by:

$$z = L' - \int_0^{L'} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

Neglecting quantities of smaller order, this reduces to:

$$z = \frac{1}{2} \int_0^{L'} \left(\frac{dy}{dx}\right)^2 \cdot dx. \tag{28}$$

If only one mode of vibration is present, $y = X \cdot \sin(\omega t)$, where $\omega/(2\pi)$ is the frequency of this mode. Substitution in (28) gives the following expression for z:

$$z = Z \sin^2 (\omega t), \tag{29}$$

where Z is given by:

$$Z = \frac{1}{2} \int_0^{L'} \left(\frac{dX}{dx}\right)^2 \cdot dx. \tag{30}$$

Equation (29) shows that the maximum longitudinal displacement is Z. As $\sin^2 (\omega t) = (1 - \cos (2\omega t))/2$, the longitudinal motion is an harmonic motion of frequency $2\omega/(2\pi)$ and amplitude $Z/2$ about a displacement $Z/2$ from the rest position. In other words, Z is the double amplitude, or full range of the longitudinal motion.

If two modes of vibration of frequencies $\omega_m/(2\pi)$ and $\omega_n/(2\pi)$ are present together, the normal displacement y is given by:

$$y = X_m \sin (\omega_m t) + X_n \sin (\omega_n t).$$

In this case, substitution in (28) shows that the longitudinal displacement z is given by:

$$z = Z_m \sin^2 (\omega_m t) + Z_n \sin^2 (\omega_n t) + Z_{mn} \sin (\omega_m t) \cdot \sin (\omega_n t), \tag{31}$$

where Z_m and Z_n are given by (29) for $X = X_m$ and $X = X_n$, respectively, and Z_{mn} is given by:

$$Z_{mn} = \int_0^{L'} \frac{(dX_m)}{(dx)} \frac{(dX_n)}{(dx)} dx. \tag{32}$$

Equation (31) shows that in this case the longitudinal displacement is

the sum of (*i*) the displacements that would be produced by each mode if it were present alone and (*ii*) an additional displacement having a maximum value of Z_{mn}. From the relation:

$$\sin(\omega_m t)\cdot\sin(\omega_n t) = \tfrac{1}{2}\left[\cos(\omega_m - \omega_n)t + \cos(\omega_m + \omega_n)t\right],$$

it follows that this additional displacement corresponds to the sum of two motions, both of amplitude $Z_{mn}/2$, having frequencies equal to the sum and difference of the frequencies of the two modes present.

In the same way, it can be shown that if there are more than two modes present, the displacement will include the displacements produced by the separate modes, and additional displacements having maximum values Z_{mn} corresponding to each pair of modes present.

To evaluate the maximum longitudinal displacements, it is necessary to obtain expressions for Z from (30) and Z_{mn} from (32). This may be done by a procedure paralleling that used by Timoshenko (loc. cit.) to derive (21). As $c^4x = d^4X/dx^4$ for any mode, $c_m^4 dX_m/dx = d^5X_m/dx^5$ and $c_n^4 dX_n/dx = d^5X_n/dx^5$. By cross multiplication and subtraction there is obtained:

$$\begin{aligned}(c_n^4 - c_m^4)\int_0^{L'} \frac{dX_m}{dx}\frac{dX_n}{dx}\,dx \\ &= \int_0^{L'}\left(\frac{dX_m}{dx}\frac{d^5X_n}{dx^5} - \frac{dX_n}{dx}\frac{d^5X_m}{dx^5}\right)\cdot dx \qquad (33)\\ &= \Big|_0^{L'}\left[\frac{dX_m}{dx}\frac{d^4X_n}{dx^4} - \frac{dX_n}{dx}\frac{d^4X_m}{dx^4} - \frac{d^2X_m}{dx^2}\frac{d^3X_n}{dx^3} + \frac{d^2X_n}{dx^2}\frac{d^3X_m}{dx^3}\right],\end{aligned}$$

where the second equation is obtained by integration by parts.

In this equation, X_m and X_n are the expressions for the deflection curve X_1 for the portion of the beam between the clamp and the prop location, corresponding to the modes of frequencies $\omega_m/(2\pi)$ and $\omega_n/(2\pi)$. Equation (33) may be used to provide an expression for Z_{mn} by substituting the boundary conditions (7), which eliminate the first two terms, and by using the derivatives of (15) to express the remaining terms. There is thus obtained:

$$\frac{L'Z_{mn}}{A_mA_n} = \frac{4\omega_m\omega_n}{\omega_n^2 - \omega_m^2}\left\{a_n\left[\frac{f_4(a_m)f_3(a_n)}{f_2(a_n)} - f_1(a_m)\right] - a_m\left[\frac{f_4(a_n)f_3(a_m)}{f_2(a_m)} - f_1(a_n)\right]\right\}, \qquad (34)$$

where $f_1(u)$, $f_2(u)$ and $f_3(u)$ are given by (16), (17), and (26), and

$f_4(u)$ is given by:

$$f_4(u) = \frac{\sin u \cdot \cosh u - \cos u \cdot \sinh u}{\cos u - \cosh u}. \tag{35}$$

An expression for Z, the integral of (30), may be obtained from (33) by considering X as a function of c and letting $c_n - c_m$ equal δc, where δc is a small quantity. Then $c_n^4 - c_m^4 = 4c_n^3 \cdot \delta c$, neglecting quantities of smaller order, and similarly, $X_n = X_m + (dX_m/dc) \cdot \delta c$. On making these substitutions in (33) and neglecting quantities of higher order of δc than the first, this equation reduces to:

$$4 \int_0^{L'} \left(\frac{dX}{dx}\right)^2 dx = {}_0^{L'}[3cXX' + c^2xX'^2 - 2c^2xXX'' - cX''X''' + c^2xX'''^2],$$

where the subscripts n and m have been dropped because, for δc negligible, c and X apply to a single mode.

As before, substitution in this of the boundary conditions (7), and of the derivatives of (15), gives finally:

$$\frac{L'Z}{A_1^2} = \frac{a}{2}\left[a + \frac{f_3(a) \cdot f_4(a)}{f_2(a)} - f_1(a)\right], \tag{36}$$

where Z is the (double) amplitude of the longitudinal motion for a single mode. By evaluating the right-hand side of (36) for particular modes and values of L'/L, and dividing these by corresponding values of $T/(m\omega^2A_1^2)$ given by (23), values are obtained of the ratio $m\omega^2L'Z/T$. Values of this ratio, determined in this way, are shown plotted in Fig. 7 against L'/L for the first five modes.

The ratios given in Fig. 7 may be used to determine the maximum longitudinal displacement for a given energy content when only one mode is present. When two or more modes are present, the maximum displacement is the sum of the displacements for the individual modes and the additional term or terms Z_{mn}. The magnitudes of these additional terms depend upon the division of the kinetic energy among the modes involved. For the case of two modes present together, the one additional term present, is, from (34), proportional to A_mA_n, and therefore to the square root of T_mT_n, the corresponding energies, whose sum $T_m + T_n$ equals the total kinetic energy T. It is easily shown that for a given value of T, T_mT_n, and therefore A_mA_n, is a maximum for $T_m = T_n = T/2$. In this case,

$$\frac{m\omega_m^2A_mA_n}{T} = \frac{\omega_m}{2\omega_n}\sqrt{\left(\frac{m\omega^2A^2}{T}\right)_m\left(\frac{m\omega^2A^2}{T}\right)_n}.$$

By evaluating the right-hand side of this equation [by means of (23)], and multiplying the result into the corresponding value of the right-hand side of (34), values may be obtained of the ratio $m\omega_m^2 L' Z_{mn}/T$. This was done for the case where $m = 1$ and $n = 2$, where the first two modes are present. The resulting values of $m\omega_1^2 L' Z_{12}/T$ are directly comparable with those of $m\omega_1^2 L' Z/T$ for the first mode alone, shown in Fig. 7. The values for $m\omega_1^2 L' Z_{12}/T$ were all less than 3 as compared with values for $m\omega_1^2 L' Z/T$ of about 50. It follows that the additional displacement resulting from the cross product term is minor.

A.10 *Angular End Displacement*

To estimate contact wipe for a spring propped at the contact, there is required an expression for the angular displacement at the prop, when this is at the end of the spring. This is given by:

$$\left(\frac{dX}{dx}\right)_L = cX_L'.$$

As in this case, $L' = L$, and $a = cL$, this equation can, by differentiation of (15), be expressed as follows:

$$\frac{L}{A_1}\left(\frac{dX}{dx}\right)_L = \frac{2a(1 - \cos a \cdot \cosh a)}{\cos a - \cosh a}. \tag{37}$$

Values of the right-hand side of this expression have been determined for the first five modes. By squaring these, and dividing them by the corresponding values of $T/(m\omega^2 A_1^2)$, there have been obtained the following values of $m\omega^2 L^2 (dX/dx)_L^2/T$.

Mode	$\dfrac{m\omega^2 L^2}{T}\left(\dfrac{dX}{dx}\right)_L^2$
1	16.416
2	49.700
3	103.68
4	178.60
5	271.44

A.11 *Use of Equations*

Many of the relations given here have been expressed in numerical form, and are shown in the figures. If additional relations are required, they may be computed from the equations given in this Appendix (or from expressions derived from them).

If such computation is required, it should be noted that all relations

involved in flexural vibrations of the type considered here can be expressed as dimensionless ratios which are, directly or indirectly, dependent on the roots of the frequency equation, and hence on the appropriate values of a, a pure number, equal to cL'. The similar number b, appearing in expressions relating to the part of the beam between the prop and the free end, is equal to $c(L - L')$, and therefore equals $a(L/L' - 1)$. The frequency, $\omega/(2\pi)$, is related to a by (6) (of which (10) is an alternate form), and the frequency ratios are therefore functions of a only. Similarly, such ratios as $F'/(sT)$, $m\omega^2 X_L^2/T$, and $m\omega^2 L'Z/T$ discussed above are all functions of a only.

Values of a have been determined for the first five modes for values of L'/L in the range from 0.5 to 1.0. To use the equations in this range and for these modes, these values of a may be easily obtained from Fig. 2, as the values of f/f_0 given there are [from (13)] equal to $(cL)^2/3.52$, so they may be used to evaluate cL, and hence $a(= cL')$.

For values of L'/L outside this range, or for modes of higher order, values of a must be determined by solution of (8) for the case in question.

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